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Coherent Population Trapping and Partial Decoherence in the Stochastic Limit

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We use the stochastic limit technique to predict a new phenomenon concerning a two-level atom with degenerate ground state interacting with a quantum field. We show, that the field drives the state of the atom to a stationary state, which is non-unique, but depends on the initial state of the system through some conserved quantities. This non uniqueness follows from the degeneracy of the ground state of the atom, and when the ground subspace is two-dimensional, the family of stationary states will depend on a one-dimensional parameter. Only one of the stationary states in this family is a pure state and it coincides with the known trapped state. This means that by controlling the initial state (input) we can control the final state (output). The quantum Markov semigroup obtained in the limit admits an invariant pure state, but it is not true that all the extremal invariant states are pure. This is an interesting phenomenon also from mathematical point of view and its meaning will be discussed in a future paper.

KEY WORDS: stochastic limit; master equation; trapped states.

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1. INTRODUCTION

In the present paper, we consider using the stochastic limit approach, a two-level atom with a degenerate ground state (or, equivalently, a three-level atom with equal energies of the two lower levels), interacting with a quantum field. This model is a generalization of the well known spin-boson model, considered by many authors. In Leggett *et al.* (1987) the dynamics of the spin-boson model was investigated with the help of functional integration (see also Caldeira *et al.*, 1981; and Leggett *et al.*, 1985 for the investigation of this model). In Accardi *et al.* (1997) an alternative derivation of the main results of Leggett *et al.* (1987) was proposed. Moreover, some new interesting regimes for the behavior of the spin-boson model were found.

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In the present paper, we find new regimes for the dynamics of a two-level atom with a degenerate ground state, interacting with a quantum field. Application of the stochastic limit approach to the dynamics of this system shows that the interaction with a quantum field drives this atom to a family of stationary states, depending on a one-dimensional parameter, which varies in an explicitly determined interval. For a particular (extremal) value of this parameter the stationary state coincides with the coherent population trapped state

$$|NC\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle) \tag{1}$$

known in quantum optics and (cf. Alzetta *et al.*, 1976; Arimondo *et al.*, 1976; Gray *et al.*, 1978 ; Arimondo, 1996; Aspect *et al.*, 1991, Harris, 1994) and also discussed in quantum information. For example, in Cirac *et al.* (1997), trapped states were used to propose a scheme to utilize photons for ideal quantum transmission between atoms located at spatially separated nodes of a quantum network. Here $|1\rangle$ and $|2\rangle$ are two orthogonal ground states of the atom. The other states in the family of stationary states, obtained in this paper, are mixtures, and can be called mixed population trapped states.

For the purposes of quantum information, it is interesting to build "decoherence free subspaces" for a given reduced evolution (cf. Zanardi *et al.*, 1997).

One way to achieve this goal is to start with an interaction Hamiltonian, whose evolution leaves these subspaces invariant.

The interaction used in the present paper does not exhibit this invariance yet some "hidden decoherence subspaces" emerge in the stochastic limit in a model which is not purely mathematical, but has many physical realizations.

Our "regime of degeneracy" is not determined only by the degeneracy of the free system Hamiltonian, but one needs in addition a more subtle condition on the generalized susceptivities, namely the degeneracy of the coefficient matrix of the right-hand side of the system (25), (26), (27), which are not present in the original Hamiltonian, but emerge only in the stochastic limit.

The above described scenario (convergence to a stationary state) can be realized in equilibrium or in a class of non equilibrium states of the quantum field. In the vacuum state new phenomena, such as quantum beats, may arise (cf. the last section).

For the investigation of the dynamics of a quantum system interacting with quantum field we use the stochastic limit approach (Accardi *et al.*, 2002). In this approach, one introduces a new slow time scale t/λ^2, where λ is a coupling constant for the interaction of the system with the field. Since in times of order t/λ^2, the interaction produces effects of order t, the described time rescaling provides a natural time scale for the observable effects of the interaction system–environment. This means that in the stochastic limit one considers effects of weak perturbations in the limit of large times. In the limit $\lambda \to 0$ the dynamics is given by Langevin and master equations, see Accardi *et al.* (2002); Accardi *et al.* (2002), which are unambiguously derived from the original Hamiltonian. For a more general mathematical discussion of Langevin and master equations see also Fagnola *et al.* (2002), Chebotarev *et al.* (2001), Antoniou *et al.* (2000).

The organization of the present paper is as follows.

In Section 2, using the stochastic golden rule, we find master equation, which describes the dynamics of the reduced density matrix of the system.

In Section 3, we analyze the system of master equations satisfied by matrix elements of the density matrix, and find different regimes of behavior for the dynamics of the system under consideration.

In Section 4, we investigate the regime of degeneracy, and prove that the system decays to one-dimensional family of stationary states.

In Section 5, we discuss the quantum beats regime, for which the quantum field, coupled to the atom, is in the Fock (or vacuum) state. In this case, instead of the decay to a stationary state, we have the decay to the regime of quantum beats inside the space of density matrices for the two lower levels.

2. THE MASTER EQUATION

We consider a three-level system with degenerate ground states $|1\rangle$, $|2\rangle$ and one excited state $|3\rangle$.

The interaction of the system with the quantum field (say, with radiation) is described by the Hamiltonian

$$H = H_S + H_R + \lambda H_I \tag{2}$$

where the system degrees of freedom are described by the Hamiltonian H_S:

$$H_S = \varepsilon_1|1\rangle\langle 1| + \varepsilon_1|2\rangle\langle 2| + \varepsilon_3|3\rangle\langle 3|$$

where ε_i is the energy of the level $|i\rangle$ (note that $\varepsilon_1 = \varepsilon_2$).

The field degrees of freedom are described by the Hamiltonian

$$H_R = \sum_i \int \omega(k)a_i^*(k)a_i(k)\, dk \tag{3}$$

where $a_i(k)$ is a boson field with a mean zero gauge invariant Gaussian state characterized by the pair correlations

$$\langle a_i^*(k)a_j(k')\rangle = N_i(k)\delta_{ij}\delta(k-k') \tag{4}$$

and $i, j = 1, 2$ are the polarization indices.

The interaction Hamiltonian H_I has the following dipole form

$$H_I = \int \sum_{i\alpha} \overline{g_{i\alpha}(k)}a_i(k)D_\alpha^* dk + \text{ h.c.} \tag{5}$$

where α takes the values 1, 2, 3, and 4, $g_{i\alpha}(k)$ are complex valued functions in the Schwartz space (the formfactors, or cut-offs), and

$$D_1 = |1\rangle\langle 3|, \qquad D_2 = |2\rangle\langle 3|, \qquad D_3 = |1\rangle\langle 2|, \qquad D_4 = |2\rangle\langle 1|$$

Remark. The last two terms of the interaction (those with coefficients D_3 and D_4) describe the transitions between the two ground states. They play a role only in the last section of the present paper (beats regime) and in fact, as clearly seen from the master equation in Section 3, their only contribution to this equation is a drift term. The most interesting (degeneracy) regime, described is Section 4 is characterized by the vanishing of the term (31), which is essentially equivalent to the vanishing of the above mentioned drift term. Thus for this regime one can suppose that $D_3 = D_4 = 0$.

The free evolution of the interaction term $a_i^{\pm}(k)D_\alpha^{\mp}$ is equivalent to an effective free evolution of the boson field of the form

$$e^{-it(\omega(k)-\omega)}a_i(k)$$

where $\omega = \varepsilon_3 - \varepsilon_1$ is the Bohr frequency, which is equal to the difference of energies of the two energy levels.

By the stochastic golden rule (Accardi *et al.*, 2002) the rescaled free evolution of the field above, in the stochastic limit, becomes a quantum white noise $b_{i\omega}(t, k)$:

$$b_{i\omega}(t, k) = \lim_{\lambda\to 0} \frac{1}{\lambda} e^{-\frac{it(\omega(k)-\omega)}{\lambda^2}} a_i(k)$$

or master field satisfying the commutation relations

$$[b_{i\omega}(t, k), b^*_{j\omega'}(t', k')] = 2\pi\delta_{\omega,\omega'}\delta_{ij}\delta(t - t')\delta(\omega(k) - \omega)\delta(k - k') \tag{6}$$

and with the mean zero gauge invariant Gaussian state with correlations:

$$\langle b^*_{i\omega}(t, k)b_{j\omega'}(t', k')\rangle = 2\pi\delta_{\omega,\omega'}\delta_{ij}\delta(t - t')\delta(\omega(k) - \omega)\delta(k - k')N_i(k) \tag{7}$$

$$\langle b_{i\omega}(t, k)b^*_{j\omega'}(t', k')\rangle = 2\pi\delta_{\omega,\omega'}\delta_{ij}\delta(t - t')\delta(\omega(k) - \omega)\delta(k - k')(N_i(k) + 1) \tag{8}$$

The Schrödinger equation becomes a white noise Hamiltonian equation, cf. Accardi *et al.* (2002), Accardi *et al.* (2002) which when put in normal order is equivalent to the quantum stochastic differential equation (QSDE)

$$dU_t = (-idH(t) - Gdt)U_t; \quad t > 0 \tag{9}$$

with initial condition $U_0 = 1$ and where

(i) $h(t)$ is the white noise Hamiltonian and $dH(t)$, called *the martingale term*, is the stochastic differential:

$$dH(t) = \int_t^{t+dt} h(s)ds = \sum_{i\alpha\omega}(D^*_\alpha dB_{i\alpha\omega}(t) + D_\alpha dB^*_{i\alpha\omega}(t)) \tag{10}$$

driven by the quantum Brownian motions

$$dB_{i\alpha\omega}(t) := \int_t^{t+dt} \int dk\overline{g}_{i\alpha}(k) b_{i\omega}(\tau, k)\, d\tau =: \int_t^{t+dt} b_{i\omega}(\tau, g_{i\alpha})\, d\tau \qquad (11)$$

(ii) The operator G, called the *drift*, is given by

$$G = \sum_{i\alpha\beta\omega} ((g_{i\alpha}|g_{i\beta})^-_\omega D^*_\alpha D_\beta + \overline{(g_{i\alpha}|g_{i\beta})^+_\omega} D_\alpha D^*_\beta) \qquad (12)$$

where the explicit form of the constants $(g_{i\alpha}|g_{i\beta})^\pm_\omega$, called the generalized susceptivities, is:

$$\begin{aligned}(g_{i\alpha}|g_{i\beta})^-_\omega &= -i \int dk\, \overline{g_{i\alpha}(k)} g_{i\beta}(k) \frac{N_i(k)+1}{\omega(k)-\omega-i0} \qquad (13)\\ &= \pi \int dk\, \overline{g_{i\alpha}(k)} g_{i\beta}(k)(N_i(k)+1)\delta(\omega(k)-\omega) - i\\ &\quad \times \text{P.P.} \int dk\, \overline{g_{i\alpha}(k)} g_{i\beta}(k) \frac{N_i(k)+1}{\omega(k)-\omega}\\ (g_{i\alpha}|g_{i\beta})^+_\omega &= -i \int dk\, \overline{g_{i\alpha}(k)} g_{i\beta}(k) \frac{N_i(k)}{\omega(k)-\omega-i0} \qquad (14)\\ &= \pi \int dk\, \overline{g_{i\alpha}(k)} g_{i\beta}(k) N_i(k)\delta(\omega(k)-\omega) - i\\ &\quad \times \text{P.P.} \int dk\, \overline{g_{i\alpha}(k)} g_{i\beta}(k) \frac{N_i(k)}{\omega(k)-\omega}\end{aligned}$$

We will use the notations

$$R\,(g_{i\alpha}|g_{i\beta})^+_\omega = \pi \int dk\, \overline{g_{i\alpha}(k)} g_{i\beta}(k) N_i(k)\delta(\omega(k)-\omega) \qquad (15)$$

$$I\,(g_{i\alpha}|g_{i\beta})^+_\omega = -\,\text{P.P.} \int dk\, \overline{g_{i\alpha}(k)} g_{i\beta}(k) \frac{N_i(k)}{\omega(k)-\omega} \qquad (16)$$

$$R\,(g_{i\alpha}|g_{i\beta})^-_\omega = \pi \int dk\, \overline{g_{i\alpha}(k)} g_{i\beta}(k)(N_i(k)+1)\delta(\omega(k)-\omega) \qquad (17)$$

$$I\,(g_{i\alpha}|g_{i\beta})^-_\omega = -\,\text{P.P.} \int dk\, \overline{g_{i\alpha}(k)} g_{i\beta}(k) \frac{N_i(k)+1}{\omega(k)-\omega} \qquad (18)$$

$$(g_{i\alpha}|g_{i\beta})^\pm_\omega = R\,(g_{i\alpha}|g_{i\beta})^\pm_\omega + i I\,(g_{i\alpha}|g_{i\beta})^\pm_\omega \qquad (19)$$

$$\overline{(g_{i\alpha}|g_{i\beta})^\pm_\omega} = R\,(g_{i\alpha}|g_{i\beta})^\pm_\omega - i I\,(g_{i\alpha}|g_{i\beta})^\pm_\omega \qquad (20)$$

Note that for $\alpha = \beta$ the values (15), (16) coincide with the real and the imaginary part of the generalized susceptivities, but, for $\alpha \neq \beta$ they are not necessarily equal to the real and imaginary parts of some number. The values in (19), (20) are related in the following way: the complex conjugate of $(g_{i\alpha}|g_{i\beta})^{\pm}_{\omega}$ is equal to $\overline{(g_{i\beta}|g_{i\alpha})^{\pm}_{\omega}}$. We also use the notation

$$(g_{\alpha}|g_{\beta})^{\pm}_{\omega} = \sum_{i}(g_{i\alpha}|g_{i\beta})^{\pm}_{\omega} \tag{21}$$

Remark 1. Typically the $g_{i\alpha}$ are matrix elements of some operators appearing in interaction Hamiltonian (cf. the description in Section 4.9.3 of Accardi *et al.* (2002)). Therefore, their dependence on the index α is often unavoidable. Therefore we will develop the theory, as far as possible, keeping this dependence explicit. In some cases, e.g., some particular classes of three-level atoms, the assumption that the formfactors $g_{i\alpha}$ do not depend on the index α, is justified. In this case the formulae simplify and are easier to interpret. This situation is described in Section 3 below.

Remark 2. Note that if the expectation $N_i(k)$ of the number operators in the reference state depends only on the dispersion $\omega(k)$ (in this case we will denote this value $N(\omega)$), then we have the identity

$$R_{\omega} = \frac{R\,(g|g)^{+}_{\omega}}{R\,(g|g)^{-}_{\omega}} = \frac{N(\omega)}{N(\omega)+1} \tag{22}$$

which shows that this quotient of the generalized susceptivities does not depend on the formfactor g. This "universality" property suggests that the quotient (22) is a natural non equilibrium generalization of the Einstein emission-absorption coefficient (cf. Section 5.9 of Accardi *et al.*, 2002).

The master equation of the stochastic limit approach for the reduced density matrix $\rho(t)$ of the system for a general discrete system with dipole interaction with a quantum field was found in Accardi *et al.* (2002), see also Accardi *et al.* (2002). The general form of this master equation is

$$\begin{aligned}\frac{d}{dt}\rho(t) = \sum_{j}\sum_{\alpha\beta}\sum_{\omega\in F}\Big(& iI(g_{j\alpha}|g_{j\beta})^{-}_{\omega}[\rho, E^{*}_{\omega}(D_{\alpha})\,E_{\omega}(D_{\beta})] \\ & - iI(g_{j\alpha}|g_{j\beta})^{+}_{\omega}[\rho, E_{\omega}(D_{\alpha})\,E^{*}_{\omega}(D_{\beta})] \\ & + 2R\,(g_{j\alpha}|g_{j\beta})^{-}_{\omega}\left(E_{\omega}(D_{\beta})\rho E^{*}_{\omega}(D_{\alpha}) - \frac{1}{2}\{\rho, E^{*}_{\omega}(D_{\alpha})E_{\omega}(D_{\beta})\}\right) \\ & + 2R\,(g_{j\alpha}|g_{j\beta})^{+}_{\omega}\left(E^{*}_{\omega}(D_{\beta})\rho E_{\omega}(D_{\alpha}) - \frac{1}{2}\{\rho, E_{\omega}(D_{\alpha})\,E^{*}_{\omega}(D_{\beta})\}\right)\Big)\end{aligned} \tag{23}$$

Here the summation on ω runs over the Bohr frequencies (energy differences for the energy levels):

$$F := \{\omega = \varepsilon_r - \varepsilon_{r'} : \varepsilon_r, \varepsilon_{r'} \in \mathrm{Spec}\, H_S\}$$

$$E_\omega(X) := \sum_{\varepsilon_r \in F_\omega} P_{\varepsilon_r - \omega} X P_{\varepsilon_r}$$

where P_{ε_r} are spectral projections for the system and

$$F_\omega := \{\varepsilon_r \in \mathrm{Spec}\, H_S : \varepsilon_r - \omega \in \mathrm{Spec} H_S\}$$
$$= \{\varepsilon_r \in \mathrm{Spec} H_S : \exists\, \varepsilon_r' \in \mathrm{Spec} H_S, \times\, \varepsilon_r - \varepsilon_r' = \omega\}$$

For the considered degenerate three-level Λ-system this equation takes the form

$$\begin{aligned}
\frac{d\rho(t)}{dt} = \sum_j \Bigg[\Bigg(& iI(g_{j1}|g_{j1})^-_\omega [\rho, |3\rangle\langle 3|] - iI(g_{j1}|g_{j1})^+_\omega [\rho, |1\rangle\langle 1|] \\
& + 2R\,(g_{j1}|g_{j1})^-_\omega \left(\rho_{33} |1\rangle\langle 1| - \frac{1}{2}\{\rho, |3\rangle\langle 3|\} \right) \\
& + 2R\,(g_{j1}|g_{j1})^+_\omega \left(\rho_{11} |3\rangle\langle 3| - \frac{1}{2}\{\rho, |1\rangle\langle 1|\} \right) \Bigg) \\
& + \Bigg(iI(g_{j2}|g_{j2})^-_\omega [\rho, |3\rangle\langle 3|] - iI(g_{j2}|g_{j2})^+_\omega [\rho, |2\rangle\langle 2|] \\
& + 2R\,(g_{j2}|g_{j2})^-_\omega \left(\rho_{33} |2\rangle\langle 2| - \frac{1}{2}\{\rho, |3\rangle\langle 3|\} \right) + 2R\,(g_{j2}|g_{j2})^+_\omega \\
& \times \left(\rho_{22} |3\rangle\langle 3| - \frac{1}{2}\{\rho, |2\rangle\langle 2|\} \right) \Bigg) + \Bigg(-iI(g_{j1}|g_{j2})^+_\omega [\rho, |1\rangle\langle 2|] \\
& + 2R\,(g_{j1}|g_{j2})^-_\omega \rho_{33} |2\rangle\langle 1| + 2R\,(g_{j1}|g_{j2})^+_\omega \left(\rho_{21} |3\rangle\langle 3| - \frac{1}{2}\{\rho, |1\rangle\langle 2|\} \right) \Bigg) \\
& + \Bigg(-iI(g_{j2}|g_{j1})^+_\omega [\rho, |2\rangle\langle 1|] + 2R\,(g_{j2}|g_{j1})^-_\omega \rho_{33} |1\rangle\langle 2| + 2R\,(g_{j2}|g_{j1})^+_\omega \\
& \times \left(\rho_{12} |3\rangle\langle 3| - \frac{1}{2}\{\rho, |2\rangle\langle 1|\} \right) \Bigg) + i \Big(I(g_{j3}|g_{j3})^-_0 [\rho, |2\rangle\langle 2|] \\
& - I(g_{j3}|g_{j3})^+_0 [\rho, |1\rangle\langle 1|] \Big) + i \Big(I(g_{j4}|g_{j4})^-_0 [\rho, |1\rangle\langle 1|] \\
& - I(g_{j4}|g_{j4})^+_0 [\rho, |2\rangle\langle 2|] \Big) \Bigg]
\end{aligned} \tag{24}$$

where, as usual $[a, b] = ab - ba$ and $\{a, b\} = ab + ba$.

One of our main results is the following separation of the density matrix into parts corresponding to invariant subspaces of the evolution.

Lemma 1. *The vector space* $H(3)$ *of the Hermitian* 3×3 *(density) matrices is the direct sum of two subspaces,* V_0, V_1, *which are invariant under the evolution, defined by (24):*

$$H(3) = V_0 \oplus V_1$$

A linear basis of V_0 *is given by* $\{|2\rangle\langle 3|, |3\rangle\langle 2|, |3\rangle\langle 1|, |1\rangle\langle 3|\}$. *Any matrix in this space decays exponentially to zero under the reduced evolution if the real parts of the generalized susceptivities (15) (for the indices* $\alpha = 1, 2$ *and* $\beta = 3$ *and vice versa) are non zero.*

A linear basis of V_1 *is given by* $\{|2\rangle\langle 1|, |1\rangle\langle 2|, |3\rangle\langle 3|, |1\rangle\langle 1|, |2\rangle\langle 2|\}$. *This space contains all the stationary states for the evolution.*

Proof. Direct verification from the right hand side of (24).

Remark 3. Notice that the space $V_1 = \mathbf{C}|3\rangle\langle 3| \oplus M$, where M is the 2×2 matrix algebra generated by $|1\rangle\langle 2|$, is itself a $*$-algebra.

Remark 4. From Lemma 1 we deduce that the evolution of the density matrix $\rho(t)$ can be split into the following sum of two evolutions $\rho_0(t)$ and $\rho_1(t)$, where $\rho_0(t)$ is an off diagonal matrix and $\rho_1(t)$ is a density matrix:

$$\rho(t) = \rho_0(t) + \rho_1(t) = \begin{pmatrix} 0 & \rho_{32}(t) & \rho_{33}(t) \\ \rho_{23}(t) & 0 & 0 \\ \rho_{13}(t) & 0 & 0 \end{pmatrix} + \begin{pmatrix} \rho_{33}(t) & 0 & 0 \\ 0 & \rho_{22}(t) & \rho_{21}(t) \\ 0 & \rho_{12}(t) & \rho_{11}(t) \end{pmatrix}$$

Moreover, $\|\rho_0(t)\| \leq e^{-ct}$, where $2c = \min R\,(g_{j\alpha}|g_{j\alpha})^{\pm}_{\omega}$, $j = 1, 2, \alpha = 1, 2$, i.e. the off-diagonal part of $\rho(t)$ (in V_0) decays exponentially whenever $c > 0$.

3. DIFFERENT REGIMES FOR THE MASTER EQUATION

In the present section we will describe the set of stationary states for the evolution, generated by the master equation (24). By Lemma 1 and Remark 4, invariant states of (24) belong to space V_1, if $\min R\,(g_{j\alpha}|g_{j\alpha})^{\pm}_{\omega} > 0$ (which is the regime of our interest). On the subspace V_1, (24) reduces to the following system of three differential equations [where we use the notations (15)–(20)]

$$\frac{d\rho_{22}(t)}{dt} = 2R\,(g_2|g_2)^{-}_{\omega}\rho_{33} - 2R\,(g_2|g_2)^{+}_{\omega}\rho_{22} - (g_1|g_2)^{+}_{\omega}\rho_{21} - \overline{(g_1|g_2)^{+}_{\omega}}\rho_{12} \quad (25)$$

$$\frac{d\rho_{11}(t)}{dt} = 2R\,(g_1|g_1)^{-}_{\omega}\rho_{33} - 2R\,(g_1|g_1)^{+}_{\omega}\rho_{11} - \overline{(g_2|g_1)^{+}_{\omega}}\rho_{21} - (g_2|g_1)^{+}_{\omega}\rho_{12} \quad (26)$$

$$\frac{d\rho_{12}(t)}{dt} = -\left(\overline{(g_1|g_1)^+_\omega} + (g_2|g_2)^+_\omega\right)\rho_{12} - (g_1|g_2)^+_\omega\rho_{11}$$
$$- \overline{(g_2|g_1)^+_\omega}\rho_{22} + 2R\,(g_2|g_1)^-_\omega\rho_{33} + i\Bigg(I(g_3|g_3)^-_0 + I(g_3|g_3)^+_0$$
$$- I(g_4|g_4)^-_0 - I(g_4|g_4)^+_0\Bigg)\rho_{12} \tag{27}$$

which together with the normalization condition

$$\rho_{11} + \rho_{22} + \rho_{33} = 1$$

the conjugation rule

$$\rho^*_{12} = \rho_{21}, \quad \rho_{11}, \rho_{22}, \rho_{33} \in \mathbf{R}$$

and the conditions of positivity of the density matrix discussed in the following Lemma, form the set of equations determining the evolution of density matrix.

Lemma 2. *The Hermitian matrix*

$$\rho = \begin{pmatrix} \rho_{33} & 0 & 0 \\ 0 & \rho_{22} & \rho_{21} \\ 0 & \rho_{12} & \rho_{11} \end{pmatrix}$$

is a density matrix iff the diagonal elements satisfy

$$\rho_{11} + \rho_{22} + \rho_{33} = 1, \qquad \rho_{11}, \rho_{22}, \rho_{33} \geq 0 \tag{28}$$

and the off-diagonal elements satisfy

$$\rho^*_{12} = \rho_{21}, \qquad |\rho_{12}|^2 \leq \rho_{11}\rho_{22} \tag{29}$$

For simplicity we discuss the case when the susceptivities $(g_\alpha|g_\beta)^\pm_\omega$, $\alpha, \beta = 1, 2$ do not depend on α, β. In this case, we denote them $(g|g)^\pm_\omega$. Introduce real variables x, y, z, t:

$$\rho_{22} - \rho_{11} = x, \qquad \rho_{21} - \rho_{12} = iy, \qquad \rho_{21} + \rho_{12} = z, \qquad \rho_{22} + \rho_{11} = t \tag{30}$$

With these notations, we get for the system (25)–(27)

$$\frac{d}{dt}x = -2R\,(g|g)^+_\omega x + 2I(g|g)^+_\omega y$$
$$\frac{d}{dt}y = -2R\,(g|g)^+_\omega y - 2I(g|g)^+_\omega x - Iz$$
$$\frac{d}{dt}z = -2R\,(g|g)^+_\omega z - 2R\,(g|g)^+_\omega t + 4R\,(g|g)^-_\omega(1-t) + Iy \tag{31}$$
$$\frac{d}{dt}t = 4R\,(g|g)^-_\omega(1-t) - 2R\,(g|g)^+_\omega t - 2R\,(g|g)^+_\omega z \tag{32}$$

where

$$I = I(g_3|g_3)_0^- + I(g_3|g_3)_0^+ - I(g_4|g_4)_0^- - I(g_4|g_4)_0^+ \tag{33}$$

The system above has the matrix form

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 4R\,(g|g)_\omega^- \\ 4R\,(g|g)_\omega^- \end{pmatrix}$$

with the matrix A equal to

$$\begin{pmatrix} -2R\,(g|g)_\omega^+ & 2I(g|g)_\omega^+ & 0 & 0 \\ -2I(g|g)_\omega^+ & -2R\,(g|g)_\omega^+ & -I & 0 \\ 0 & I & -2R\,(g|g)_\omega^+ & -2R\,(g|g)_\omega^+ - 4R\,(g|g)_\omega^- \\ 0 & 0 & -2R\,(g|g)_\omega^+ & -2R\,(g|g)_\omega^+ - 4R\,(g|g)_\omega^- \end{pmatrix} \tag{34}$$

For the determinant of A we get

$$\text{Det}\ A = 2R\,(g|g)_\omega^+ \left(2R\,(g|g)_\omega^+ + 4R\,(g|g)_\omega^-\right) I^2$$

Since, if the determinat is nonzero, there is no zero eigenvalues, therefore the system of equations in this case has no nontrivial stationary states.

We see that the behavior of the density matrix is described by the value I, given by (33), which arises from the coupling of two lower levels (which have the same energy). We will say that these lower levels are uncoupled, if the value I defined by (33) is zero. In the uncoupled regime the transitions between the two lower levels $|1\rangle$ and $|1\rangle$ compensate, and the contribution to the master equation corresponding to these transitions vanish. In this case the determinant of the matrix A is zero and therefore the system (25)–(27) has a family of stationary states. In this case, see in the next section for the details, the stationary states are degenerate and form one dimensional family, and the dynamics of the density matrix is described by exponential decay to the family of stationary states (if $R\,(g|g)_\omega^+ > 0$). We call this regime (when $I = 0$) the regime of degeneracy.

If the value (33) is nonzero, then Det A is positive (if $R\,(g|g)_\omega^+ > 0$), and we obtain that the density matrix will converge to the unique stationary state.

4. THE REGIME OF DEGENERACY

Let us start with the investigation of the degeneracy regime, when the value (33) is zero.

Lemma 3. *When the susceptivities $(g_\alpha|g_\beta)_\omega^\pm$, $\alpha, \beta = 1, 2$ do not depend on α, β (in this case, we denote them $(g|g)_\omega^\pm$) the system (25)–(27) of linear equations* *determining the evolution of the atom has the conservation law*

$$\rho_{11}(t) + \rho_{22}(t) = \rho_{12}(t) + \rho_{21}(t) + C; \qquad \forall t \tag{35}$$

Moreover, if $R(g|g)_\omega^- > 0$*, then* $\rho_{12}(t) + \rho_{21}(t)$ *converges exponentially in time to the stationary value*

$$\rho_{12} + \rho_{21} = \frac{4R\,(g|g)_\omega^-(1 - C) - 2R\,(g|g)_\omega^+ C}{4(R\,(g|g)_\omega^- + R\,(g|g)_\omega^+)} = \frac{1 - C - CR_\omega/2}{1 + R_\omega} \tag{36}$$

and C *is the real constant. Here* R_ω *is defined by (22).*

The constant C has the form

$$C = 1 - \rho_{33} - 2\,\mathrm{Re}\,\rho_{12}$$

Proof. Since the value (33) is 0, comparing (31) and (32) and using (30), one easily deduces from the identity

$$\frac{d\rho_{11}(t)}{dt} + \frac{d\rho_{22}(t)}{dt} = \frac{d\rho_{12}(t)}{dt} + \frac{d\rho_{21}(t)}{dt}$$

From which, the conservation law (35) follows. It implies that for fixed C, the evolution of the system is characterized by the real function of time $\rho_{12}(t) + \rho_{21}(t)$ which we denote $2s(t)$:

$$s(t) = \frac{1}{2}\,(\rho_{12}(t) + \rho_{21}(t))$$

With this notation the system (25)–(27) implies that

$$\frac{ds(t)}{dt} = -4\left(R\,(g|g)_\omega^- + R\,(g|g)_\omega^+\right) s(t) + 2R\,(g|g)_\omega^-(1 - C) - R\,(g|g)_\omega^+ C \tag{37}$$

If $R(g|g)_\omega^- > 0$, then (37) implies the exponential decay of $s(t)$ to the stationary value (36) and this proves the lemma.

Remark 5. Note that the condition $R(g|g)_\omega^- > 0$ means that

$$\int \overline{g(k)}g(k)\delta(\omega(k) - \omega)dk \neq 0$$

which is automatically satisfied when the support of the formfactor $g(k)$ intersects the resonant surface $\omega(k) = \omega$ in a set of nonzero (surface) measure.

The stationary solution of the system (25)–(27) is determined by the system of equations

$$\begin{aligned} 2R\,(g_2|g_2)_\omega^-(\rho_{11} + \rho_{22}) + 2R\,(g_2|g_2)_\omega^+\rho_{22} + (g_1|g_2)_\omega^+\rho_{12}^* \\ + \overline{(g_1|g_2)_\omega^+}\rho_{12} = 2R\,(g_2|g_2)_\omega^- \end{aligned} \tag{38}$$

$$2R\,(g_1|g_1)^-_\omega(\rho_{11}+\rho_{22})+2R\,(g_1|g_1)^+_\omega\rho_{11}+\overline{(g_2|g_1)^+_\omega}\rho^*_{12} + (g_2|g_1)^+_\omega\rho_{12}=2R\,(g_1|g_1)^-_\omega \tag{39}$$

$$\left(\overline{(g_1|g_1)^+_\omega}+(g_2|g_2)^+_\omega\right)\rho_{12}=-(g_1|g_2)^+_\omega\rho_{11}-\overline{(g_2|g_1)^+_\omega}\rho_{22}+2R\,(g_2|g_1)^-_\omega\rho_{33} \tag{40}$$

Remark 6. For different formfactors $g_\alpha(k)$ the system (25)–(27) may have different behavior. In the generic case for $g_1\neq g_2$ the stationary solution is unique. For instance when g_1 is orthogonal to g_2 (in the sense of the bilinear form $(g_1|g_2)^+_\omega$), then the determinant of the system (38), (39) reduces to

$$-\left(2R\,(g_2|g_2)^+_\omega 2R\,(g_1|g_1)^-_\omega+2R\,(g_2|g_2)^-_\omega 2R\,(g_1|g_1)^+_\omega+2R\,(g_2|g_2)^+_\omega 2R\,(g_1|g_1)^+_\omega\right)$$

and whenever this determinant is nonzero, the solution is unique.

When $g_1=g_2$ the solution of the system above is nonunique due to Lemma 3.

Now we are ready to formulate the following theorem describing the structure of the stationary density matrices.

Theorem 4. *For $(g_\alpha|g_\beta)^\pm_\omega$ not depending on α, β and when*

$$R\,(g|g)^+_\omega>0 \tag{41}$$

the system of linear (38)–(40) determining the stationary state of the atom possesses a family of solutions parameterized by the one-dimensional parameter s:

$$\rho=\begin{pmatrix}\rho_e & 0 & 0\\ 0 & \rho_g & s\\ 0 & s & \rho_g\end{pmatrix} \tag{42}$$

where, in the notation (22)

$$\rho_e=\frac{2R\,(g|g)^+_\omega(1+2s)}{4R\,(g|g)^-_\omega+2R\,(g|g)^+_\omega}=\frac{(1+2s)R_\omega}{2+R_\omega} \tag{43}$$

$$\rho_g=\frac{2R\,(g|g)^-_\omega-2R\,(g|g)^+_\omega s}{4R\,(g|g)^-_\omega+2R\,(g|g)^+_\omega}=\frac{1-sR_\omega}{2+R_\omega} \tag{44}$$

The admissible values of the parameter s are precisely those for which

$$\frac{1}{2(1+R_\omega)}=\frac{1}{2}\left(1+\frac{R\,(g|g)^+_\omega}{R\,(g|g)^-_\omega}\right)^{-1}\geq s\geq-\frac{1}{2} \tag{45}$$

Moreover, if (41) is satisfied, the solution of the system (25)–(27) converges exponentially, as $t\to\infty$, to the stationary state (42).

Proof. If $g_1=g_2=g$, then (38)–(40) take respectively the form:

$$2R\,(g|g)^-_\omega(\rho_{11}+\rho_{22})+2R\,(g|g)^+_\omega\rho_{22}+(g|g)^+_\omega\rho_{21}+\overline{(g|g)^+_\omega}\rho_{12}=2R\,(g|g)^-_\omega \tag{46}$$

$$2R\,(g|g)^-_\omega(\rho_{11}+\rho_{22})+2R\,(g|g)^+_\omega\rho_{11}+\overline{(g|g)^+_\omega}\rho_{21}+(g|g)^+_\omega\rho_{12}=2R\,(g|g)^-_\omega \tag{47}$$

$$2R\,(g|g)^+_\omega\rho_{12}=-(g|g)^+_\omega\rho_{11}-\overline{(g|g)^+_\omega}\rho_{22}+2R\,(g|g)^-_\omega\rho_{33} \tag{48}$$

Taking the differences of (46), (47) and of (48) and its conjugate, we obtain

$$2R\,(g|g)^+_\omega(\rho_{22}-\rho_{11})+2iI\,(g|g)^+_\omega(\rho_{21}-\rho_{12})=0$$
$$2R\,(g|g)^+_\omega(\rho_{21}-\rho_{12})+2iI\,(g|g)^+_\omega(\rho_{22}-\rho_{11})=0$$

Taking the sum of the two equations above and dividing by two, we get

$$(g|g)^+_\omega(\rho_{22}-\rho_{11}+\rho_{21}-\rho_{12})=0$$

If $(g|g)^+_\omega\neq 0$, then since $\rho_{22}-\rho_{11}$ is real, and $\rho_{12}-\rho_{21}$ is imaginary, we obtain

$$\rho_{22}=\rho_{11},\qquad \rho_{21}=\rho_{12} \tag{49}$$

In particular, ρ_{12} must be a real number. Then, the sum of (46) and (47) takes the form

$$2R\,(g|g)^+_\omega(\rho_{11}+\rho_{22}+\rho_{12}+\rho_{21})=4R\,(g|g)^-_\omega\rho_{33} \tag{50}$$

Equations (49) and (50) imply that any stationary density matrix must satisfy the following condition:

$$2R\,(g|g)^+_\omega(\rho_{11}+\rho_{12})=2R\,(g|g)^-_\omega\rho_{33} \tag{51}$$

Since under the condition (49), the (46) and (47) coincide, the (49), (51) describe the general stationary solution for (25)–(27). From these (49), (51) and (28), we obtain

$$\rho_{11}=\rho_{22}=\frac{2R\,(g|g)^-_\omega-2R\,(g|g)^+_\omega\rho_{12}}{4R\,(g|g)^-_\omega+2R\,(g|g)^+_\omega}=\frac{1-sR_\omega}{2+R_\omega}$$
$$\rho_{33}=\frac{2R\,(g|g)^+_\omega+4R\,(g|g)^+_\omega\rho_{12}}{4R\,(g|g)^-_\omega+2R\,(g|g)^+_\omega}=\frac{1+2R_\omega s}{2+R_\omega} \tag{52}$$

From (52) and (29) one sees that the positivity of the density matrix is equivalent to inequalities

$$\frac{1}{2}\left(1+\frac{R\,(g|g)^+_\omega}{R\,(g|g)^-_\omega}\right)^{-1}\geq\rho_{12}\geq-\frac{1}{2} \tag{53}$$

Conversely, taking any real value of ρ_{12} satisfying (53) and determining ρ_{11} and ρ_{33} by (52), one obtains a stationary state for the master equations (25)–(27).

Let us now prove exponential convergence of the system to a stationary state. The system (25)–(27) implies

$$\frac{d}{dt}(\rho_{22}-\rho_{11}) = -2R\,(g|g)^+_\omega(\rho_{22}-\rho_{11}) + 2iI(g|g)^+_\omega(\rho_{12}-\rho_{21}) \qquad (54)$$

$$\frac{d}{dt}(\rho_{12}-\rho_{21}) = -2R\,(g|g)^+_\omega(\rho_{12}-\rho_{21}) + 2iI(g|g)^+_\omega(\rho_{22}-\rho_{11}) \qquad (55)$$

Adding these two equations we see that

$$\rho_{22}-\rho_{11}+\rho_{12}-\rho_{21} = \text{ const } e^{t\left(-2R\,(g|g)^+_\omega+2iI(g|g)^+_\omega\right)} \qquad (56)$$

and, if $R\,(g|g)^+_\omega > 0$ the linear combination (56) converges exponentially to zero. Since $\rho_{22}-\rho_{11}$ is real and $\rho_{12}-\rho_{21}$ is imaginary, we obtain that (56) converges to the state where $\rho_{22}=\rho_{11}$ and $\rho_{12}=\rho_{21}$ (and therefore is real).

Then, applying Lemma 3, we get that ρ_{33} converge to stationary values, which are controlled by the stationary value $s=\frac{1}{2}(\rho_{12}+\rho_{21})$.

This finishes the proof of the theorem.

Since the generalized susceptivities are given by the expression

$$R\,(g_i|g_i)^+_\omega = \pi\int |g_i(k)|^2 N_i(k)\delta(\omega(k)-\omega)\,dk$$

$$R\,(g_i|g_i)^-_\omega = \pi\int |g_i(k)|^2 (N_i(k)+1)\delta(\omega(k)-\omega)\,dk$$

$$R\,(g|g)^\pm_\omega = \sum_i R\,(g_i|g_i)^\pm_\omega$$

It follows that one has inequality

$$R\,(g|g)^-_\omega > R\,(g|g)^+_\omega$$

One can see that for high intensity of radiation, i.e., when $N_i(k)\gg 1$, one can put $R_\omega = \frac{N_i(k)}{N_i(k)+1} = 1$. In this case the solution (42), (45) will be simplified as follows

$$\rho = \begin{pmatrix} \frac{1+2s}{3} & 0 & 0 \\ 0 & \frac{1-s}{3} & s \\ 0 & s & \frac{1-s}{3} \end{pmatrix}, \qquad \frac{1}{4} \ge s \ge -\frac{1}{2}$$

The most interesting states correspond to the extremal values of the parameter ρ_{12}. The minimal value of ρ_{12} is $-\frac{1}{2}$, which correspond to the density matrix for the pure state $|NC\rangle$:

$$\rho\text{min} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} = |NC\rangle\langle NC| = \frac{1}{2}\left(|1\rangle\langle 1| + |2\rangle\langle 2| - |1\rangle\langle 2| - |2\rangle\langle 1|\right)$$

where the vector

$$|NC\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$$

is exactly the coherent population trapped state (1), discussed in the literature Alzetta *et al*. (1976); Arimondo *et al*. (1976); Gray *et al*. (1978), Arimondo (1996); Aspect *et al*. (1991). In the same approximation the maximal value $\rho_{12} = \frac{1}{4}$ corresponds to the density matrix

$$\rho_{\max} = \frac{1}{4}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \frac{1}{2}|3\rangle\langle 3| + \frac{1}{2}|C\rangle\langle C|$$

This state is mixed, but the state of the reduced system corresponding to levels $|1\rangle$ and $|2\rangle$ is pure with the state vector

$$|C\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$$

(called the coupled state in Alzetta *et al*., (1976); Arimondo *et al*. (1976); Gray *et al*. (1978)).

Remark 7. To distinguish experimentally different stationary states from the obtained family, one can measure the following observable

$$A = |1\rangle\langle 2| + |2\rangle\langle 1|$$

which for instance may describe the interaction of the hyperfine split levels with a magnetic field.

In fact the measurement of A in the stationary state gives

$$\mathrm{tr}\, \rho A = \rho_{12} + \rho_{21} = 2s$$

and the different stationary states give different mean values of A.

Remark 8. (44), (45) imply that the minimum ground state population is achieved when the parameter s is maximum, i.e.,

$$\rho_g^{\min} = \frac{1}{2 + R_\omega}\left(1 - \frac{R_\omega}{2 + 2R_\omega}\right) = \frac{1}{2}\frac{1}{1 + R_\omega} = \frac{1}{2}\frac{N(\omega) + 1}{2N(\omega) + 1} > \frac{1}{4}$$

Since the ground level population is $2\rho_g$, it follows that in any stationary state at least $1/2$ of the population is in the ground level. On the other hand the above chain of identities shows, in that the region of high radiation intensity $N(\omega) \gg 1$, the estimate $\rho_g^{\min} = \frac{1}{4}$ is almost exact and therefore we can conclude that, in this region, for any stationary state, the population of the excited state is about $1/2$.

Remark 9. If for $\alpha, \beta = 1, 2$

$$(g_\alpha | g_\beta)_\omega^+ = 0$$

in particular, in the Fock case, the stationary solutions of (25)–(27) (neglecting the trivial case when also $(g_\alpha|g_\beta)^-_\omega = 0$) is characterized by the single condition

$$\rho_{11} + \rho_{22} = 1, \quad \rho_{11}, \rho_{22} \geq 0$$

so that $\rho_{33} = 0$ and ρ_{12} is arbitrary and subject only to the constraints (29).

Remark 10. Note that if $R\,(g|g)^+_\omega = 0$ and $I\,(g|g)^+_\omega \neq 0$ then (56) implies that the system does not converge to a stationary state but has an oscillatory behavior.

5. QUANTUM BEATS REGIME

Consider the evolution of the sytem under investigation in the Fock state, when all $(g|g)^+$ are equal to zero. In this regime there exists possibility that there is no decay to the stationary state and we have the oscillations. We consider again the case when the susceptivities $(g_\alpha|g_\beta)^\pm_\omega$ do not depend on α, β.

In the considered case the matrix (34) takes the form

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 \\ 0 & I & 0 & -4R\,(g|g)^-_\omega \\ 0 & 0 & 0 & -4R\,(g|g)^-_\omega \end{pmatrix}$$

By Remark 5 after Lemma 3 it is natural to assume that $R\,(g|g)^-_\omega > 0$. In this case the off-diagonal matrix elements ρ_{13}, ρ_{23} decay exponentially by Remark 4, cf. Accardi *et al*. (2002).

Analyzing the system equations for the density matrix, one can check that in the considered case the long time dynamics is described by

$$\rho_{13} = \rho_{23} = 0, \quad \rho_{22} + \rho_{11} = 1, \quad \rho_{33} = 0, \quad \rho_{22} - \rho_{11} = \text{ const}$$
$$\rho_{12} = \text{ const } e^{itI} \tag{57}$$

where ρ_{ij} are complex numbers satisfying Lemma 2.

This kind of pure oscillatory behavior without damping is related to the quantum beats.

6. CONCLUSION

In the present paper we investigated the interaction of an atom with a degenerate ground state with a quantum field. We find (under natural conditions for the formfactors), that the evolution drives the atom exponentially to a stationary state. This stationary state is not unique, and the family of stationary states may be parameterized by a one-dimensional parameter. For a special (minimal) value of this parameter the obtained stationary state is pure and coincides with the population trapped state, known in the literature Alzetta *et al.* (1976); Arimondo *et al.* (1976); Gray *et al.* (1978), Arimondo, 1996, Aspect *et al.* (1991). The obtained results show the possibility of emergence of mixed stationary states, which continuously interpolate between the coupled and the non-coupled states. This difference can be experimentally detected.

In the case of special states (say the Fock state) also the oscillatory behavior (57) is possible.

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REFERENCES

Accardi, L., Kozyrev, S. V., and Volovich, I. V. (1997). Dynamics of dissipative two-level system in the stochastic approximation. *Physsical Review A* **56**, 2557–2562, http://xxx.lanl.gov/abs/quant-ph/9706021.

Accardi, L., Lu, Y. G., and Volovich, I. V. (2002). *Quantum theory and its stochastic limit*, Springer Verlag, New York.

Accardi, L., and Kozyrev, S. V. (2002) Lectures on quantum interacting particle systems. In QP-PQ: Quantum Probability and White Noise Analysis, *Quantum Interacting Particle Systems*, Vol. 14, World Scientific Publishing.

Alzetta, G., Gozzini, A., Moi, L., and Orriols, G. (1976). *Nuovo Cimento* **B36**, 5; Arimondo, E. and Orriols, G. (1976). *Nuovo Cimento Letter* **17**, 333; Gray, H. M., Whitley, R. M., and Stroud C. R. (1978). *Journal of Optical Letter* **3**, 218.

Antoniou, I. and Smolyanov, O. G. (2000). Feynman and Wiener path integral representations for Liouville evolution, In *Evolution Equations and their Applications in Physical and Life Sciences, Lecture Notes in Pure and Applied Mathematics*, Marcel Dekker,[Location?], 215.

Arimondo, E. (1996). Relaxation processes in coherent population trapping. *Physical Review A* **54**, 2216.

Aspect, A., Arimondo, E., Kaiser, R., Vansteenkiste, N. and Cohen-Tannoudji, C. (1991). Laser cooling below the one-photon recoil energy by velocity-selective coherent population trapping. *Physical Review Letters* **61**, 826.

Caldeira, A. O. and Leggett, A. J. (1981). Influence of Dissipation on Quantum Tunneling in Macroscopic Systems. *Physical Review Letters* **46**, 211–214.

Chebotarev, A., Garcia, J., and Quezada, R. (2001). Interaction representation method for Markov master equations in quantum optics, http://xxx.lanl.gov/abs/quant-ph/0106094. In *Stochastic*

Analysis and Mathematical Physics, ANESTOC, Proceeding of the Fourth International Workshop, Birkhauser, Boston.

Cirac, J. I., Zoller, P., Kimble, H. J., and Mabuchi, H. (1997). Quantum state transfer and entanglement distribution among distant nodes in a quantum network. *Physical Review Letters* **78**, 3221.

Fagnola F., and Rebolledo R. (2002). Lectures on the qualitative analysis of quantum markov semigroups, Cirm-Volterra International School Quantum Interacting Particle Systems Levico Terme. In Accardi L., Fagnola F. editors. *Quantum Interacting Particle Systems, $QP - PQ$ Quantum Probability and White Noise Analysis XIV*, World Scientific.

Harris, S. E. (1997). Electromagnetically induced transparency. *Physics Today*, 36–42.

Harris, S. E. (1994) Normal modes for electromagnetically induced transparency, *Physical Review Letters* **72**, 52.

Leggett, A. J., Chakravarty, S., Dorsey, A. T., Fisher, M. P. A., Garg, A., and Zwerger, W. (1987) Dynamics of the dissipative two-state system, *Reviews of Modern Physics* **59**, 1–85.

Leggett, A. J., and Anupam Garg. (1985). Quantum mechanics versus macroscopic realism: Is the flux there when nobody looks? *Physical Review Letters* **54**, 857–860.

Zanardi P., Rasetti M. (1997). Noiseless quantum codes. *Physical Review Letters* **79**, 17.